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On the Statistical Usefulness and Application of Sample Range[1]

by

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Abstract

We gave so far the g.l.b. and the l.u.b. of the second moment centered at the mean, so called s.s., by using the range R, and similar basic formulae for various data structures mathematically. We also devised the ways to prove them. In order to investigate the relation statistically, we give a table of correction multipliers of R to give unbiased estimates of the population standard deviation σ, a table of efficiency of estimation by R compared with estimation by the sample standard deviation s, etc. Thereafter illustrative examples by practical data and a consideration to effective usage of the range R are given

要　約

これまでデータのレインジ R を用いて平均値のまわりの変動量 s.s. の上限, 下限を数学的に求め, さまざまなデータ構造に対しても, 同様な基本的形式を示し, その証明方法にも工夫を加えた.

この論文ではその関係を統計的に考察するために, 母標準偏差 σ の不偏推定値を求めるレインジ R の補正係数の数表や, 標本標準偏差 s との効率比較の数表などを求めている. 更に, 実践データによるその検証も行い, レインジ R の有効利用の考察を行っている.

Key Words： sample range, unbiased estimator, efficiency of estimations

キーワード： 標本のレインジ, 不偏推定量, 推定の効率

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§ 1 . Introduction

The theme of our study and the developments.

After G. Snedecor [12], the usage of range estimator had been studied in the area of quality control under the production process.

We have studied the mathematical relation between R and s^2 for data $x_1, x_2, \ldots, x_n$. Especially we gave the l.u.b. and the g.l.b. of s^2 by using R mathematically in T. Mendori et al. [4].

Furthermore, we gave the relation for the mixed and the classified data set in [5], [6], [7].

In this paper we present the following :

1. We further investigate the validity of R to unbiased estimation of σ, and the approach by the mean of ranges : $\overline{R} = \frac{1}{k}\sum_{i=1}^{k} R_i$ in § 2 and § 4.
2. We give tables of numerical calculations for the related statistics in § 3.
3. We investigate the possibility to introduce the mathematical relation of R and s^2 to the statistical inference and give the TABLE 5.
4. We suggest applying these studies to practical data in § 4 and its possible development in § 5.

The comments on the range by G. Snedecor served as a clue to our study. Our interest is purely mathematical problem to evaluate the s.s. of n data, $g(\overline{x}) = \sum_{i=1}^{n}(x_i - \overline{x})^2$, by using the range $R = \max x_i - \min x_i$. Our results shows that the g.l.b. is independent of the data number n, while the l.u.b. does not exceed $\frac{n}{4} R^2$, regardless of the oddity of n. On this relation we gave various ways to prove it in Mendori et al. [4], [7].

On the usability of the range R as a substitute of the sample s.d. s or that of the unbiased estimator u of the population s.d. σ, the statistical investigation about validity and efficiency was carried out in the area of quality control with the necessity of sampling inspection.

In the present-day computing environment where we can calculate precisely and quickly, the idea of substitution is not taken seriously. But the investigation of maintainable and reliable substitute is useful, and some people still consider the simplicity valuable. For example, could we stop production process, if a trouble in a computer occured?

We introduce the results of our study to the statistical consideration, and sum up the possibility and limit of the substitution again.

Both the issue of global warming pointed out in the book "An Inconvenient Truth" by Al Gore and the national student achievement assessment are related to the variation of deviation, and for promoting of statistical literacy these kinds of study might be useful. Especially it is interesting to relate mathematical formulae to statistical discussion.

§ 2. Statistical discussions

On transforming random samples X_1, X_2, ..., X_n to order statistics Y_1, Y_2..., Y_n, $Y_1 \leq Y_2 \leq ... \leq Y_n$, the density function of the simultaneous distribution $g(y_1, y_2, ..., y_n)$ is given by

$$g(y_1, y_2, \cdots, y_n) = n! f(y_1) f(y_2) \cdots f(y_n),$$

where each of X_i has an independently identically distributed density function $f(x)$. Then the probability density function $g(R)$ of the sample range $R = Y_n - Y_1$ is written by

$$g(R) = n(n-1) \int_{-\infty}^{\infty} \left\{ \int_{y_1}^{R+y_1} f(y) dy \right\}^{n-2} f(y_1) f(R+y_1) dy_1 .$$

In the case the function $f(x)$ is the density of the normal distribution, we can find a constant b_n such that

$$E(b_n R) = \sigma .$$

Namely, an unbiased estimator $b_n R$ of the population standard deviation σ is given by a coefficient b_n, which depends on the sample number n. Since we have

$$\begin{aligned} V(b_n R) &= E\{(b_n R)^2\} - \{E(b_n R)\}^2 \\ &= b_n^2 E(R^2) - b_n^2 E(R)^2 = b_n^2 \{E(R^2) - E(R)^2\} \\ &= b_n^2 V(R), \end{aligned}$$

the standard error $D(b_n R)$ of the estimator $b_n R$ is given as follows :

$$D(b_n R) = \sqrt{V(b_n R)} = |b_n|\sqrt{V(R)}\,.$$

Although $u^2 = \dfrac{n}{n-1}s^2$ is known to be an unbiased estimator of σ^2, $u = \sqrt{\dfrac{n}{n-1}s^2}$ is not an unbiased estimator of σ. So, similar to the case of R, we can find another constant a_n such that $a_n\, s$ is an unbiased estimator of the population standard deviation σ, i.e.,

$$E(a_n\, s) = \sigma,$$

where s is the sample standard deviation :

$$s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2}\,.$$

For a sample with n independently identically distributed normal variables $X_i \sim N(\mu,\, \sigma^2)$, the statistic $\dfrac{ns^2}{\sigma^2}$ has a chi-square distribution with $(n-1)$ degrees of freedom, since it holds

$$n\,s^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2\,.$$

It follows that

$$E\left(\frac{\sqrt{n\,s^2}}{\sigma}\right) = E\left(\sqrt{n}\cdot\frac{s}{\sigma}\right) = \sqrt{2}\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)},$$

then we have

$$E(s) = \sqrt{\frac{2}{n}}\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\,\sigma,$$

where the symbol Γ denotes Gamma-function. Therefore using the coefficient a_n given by

$$a_n = \sqrt{\frac{n}{2}}\cdot\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)},$$

the statistic $a_n\ s$ is an unbiased estimator of σ: $E(a_n\ s) = \sigma$.

In order to propose using $b_n\ R$ as an substitute of $a_n\ s$ for an unbiased estimator of σ, let us define an efficiency of R to s by the following:

$$\eta(n) = \frac{V(a_n\, s)}{V(b_n\, R)} = \left\{ \frac{D(a_n\, s)}{D(b_n\, R)} \right\}^2 .$$

By preparing a numerical table of n and $\eta\,(n)$, the level of efficiency can be discussed. Compared to s the calculation of R needs only the maximum and the minimum value of n data and needs no average. This handiness reduces the efficiency, but the utilization can be considered by the tolerable number n.

Furthermore, when the whole data is divided into k groups with n data, namely, $N = kn$, the estimation error for the mean value $\overline{R} = \frac{1}{k}\sum_{i=1}^{k} R_i$ of the ranges R_i, $i = 1, 2, \ldots, k$ is known to be $\frac{1}{\sqrt{k}}$ times of the estimation error for the whole range R.

$$D(b_n\, \overline{R}) = \sqrt{V(b_n\, \overline{R})} = \frac{1}{\sqrt{k}} \sqrt{V(b_n\, R)}.$$

On the other hand, since it holds approximately as N is increasing

$$D(a_N\, s) = \sqrt{V(a_N\, s)} \approx \frac{1}{\sqrt{2kn}} \sqrt{V(a_n\, s)},$$

we have

$$\frac{D(b_n\, \overline{R})}{D(a_N\, s)} \approx \sqrt{2n}\, \sqrt{V(b_n\, R)}.$$

By checking the TABLE 4, the standard error attains minimum at $n = 8$.

Concerning the inequality given by our Main Theorem [4 ; Theorem 2.1], the same inequality holds for the mean (i.e. expectation) :

$$\frac{1}{\sqrt{2}} E(R) \leq \sqrt{n}\, E(s) \leq \sqrt{\frac{n}{4} - \frac{1-(-1)^n}{8n}}\, E(R).$$

The inequality holds for any independent identical distribution of X_i, $i = 1, 2, \ldots,$ n, we give a table for the normal distribution.

§ 3. Calculations and its tables for the related statistics

< TABLE 1 > An unbiased estimator of σ by using R.

We have

$$E(b_n R) = \sigma\,,$$
$$V(b_n R) = E\left\{(b_n R)^2\right\} - \{E(b_n R)\}^2$$
$$= b_n^2 V(R)\,,$$

Let d_2, d_3 be multipliers of such that

$$E(R) = d_2\,\sigma\,,$$

and

$$D(R) = \sqrt{V(R)} = d_3\,\sigma.$$

then we have

$$d_3 = \frac{\sqrt{V(b_n R)}}{b_n} = d_2\sqrt{V(b_n R)}\,.$$

Under the assumption of normal distribution we give a table of modification multiplier b_n of R for the population standard deviation σ. For $n = 2$ or 3, we immediately have

$$b_2 = \frac{\sqrt{\pi}}{2}, \qquad b_3 = \frac{\sqrt{\pi}}{3}$$

by calculating the defining integral. But for $n \geq 4$, we need to use the numerical integration method. We also give the variance $V(b_n R)$ and d_2, d_3 together in TABLE 1. In the following tables the numbers are rounded off to the fourth decimal place.

TABLE 1. Multiplier b_n and the variance of b_n R, etc.

n	b_n	$V(b_n R)$	d_2	d_3
2	0.8862	0.5708	1.1284	0.8525
3	0.5908	0.2755	1.6926	0.8884
4	0.4857	0.1826	2.0588	0.8798
5	0.4299	0.1380	2.3259	0.8641
6	0.3946	0.1120	2.5344	0.8480
7	0.3698	0.0949	2.7044	0.8332
8	0.3512	0.0829	2.8472	0.8198
9	0.3367	0.0740	2.9700	0.8078
10	0.3249	0.0671	3.0775	0.7971
20	0.2677	0.0381	3.7349	0.7287
25	0.2544	0.0325	3.9306	0.7084
30	0.2448	0.0287	4.0855	0.6927
50	0.2223	0.0210	4.4981	0.6522
100	0.1994	0.0146	5.0152	0.6052

< TABLE 2 >

Let a_n, $a_n{}^*$ be multipliers of s, u, respectively, such that

$$E(a_n\, s) = \sigma, \quad E(a_n{}^*\, u) = \sigma,$$

where

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2, \quad u^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Since

$$s = \sqrt{\frac{n-1}{n}}\,u, \quad E(a_n\, s) = E\left(a_n\sqrt{\frac{n-1}{n}}\,u\right) = E(a_n{}^*\, u),$$

we have

$$a_n{}^* = \sqrt{\frac{n-1}{n}}\,a_n = \sqrt{\frac{n-1}{2}}\cdot\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

For example, we have

$$a_2{}^* = \frac{1}{\sqrt{2}}\cdot\Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{2}} \approx 1.2533,$$

$$a_3^* = \frac{1}{\Gamma\left(\frac{3}{2}\right)} = \frac{2}{\Gamma\left(\frac{1}{2}\right)} = \frac{2}{\sqrt{\pi}} \approx 1.1284,$$

$$a_{10}^* = \frac{3}{\sqrt{2}} \cdot \frac{\Gamma\left(4+\frac{1}{2}\right)}{\Gamma(5)} \approx 1.0281.$$

TABLE 2. Multiplier a_n and the variance of a_n s, etc.

n	a_n	$V(a_n\ s)$	a_n^*	$V(a_n^*\ u)$
2	1.7725	0.5708	1.2533	0.5708
3	1.3820	0.2732	1.1284	0.2732
4	1.2533	0.1781	1.0854	0.1781
5	1.1894	0.1318	1.0638	0.1318
6	1.1512	0.1045	1.0509	0.1045
7	1.1259	0.0865	1.0424	0.0865
8	1.1078	0.0738	1.0362	0.0738
9	1.0942	0.0643	1.0317	0.0643
10	1.0837	0.0570	1.0281	0.0570
20	1.0396	0.0267	1.0132	0.0267
30	1.0259	0.0174	1.0087	0.0174
50	1.0153	0.0103	1.0051	0.0103
100	1.0076	0.0051	1.0025	0.0051

Similarly we have

$$\begin{aligned} V(a_n^*\,u) &= E\left\{(a_n^*\,u)^2\right\} - \{E(a_n^*\,u)\}^2 \\ &= (a_n^*)^2\,E(u^2) - \sigma^2 \\ &= \left\{(a_n^*)^2 - 1\right\}\sigma^2, \end{aligned}$$

where $E(u^2) = \sigma^2$.

Note.

$$E(u) = \frac{1}{a_n^*}\sigma = d_2^*\,\sigma,$$

$$V(u) = (d_3^*)^2\,\sigma^2 = \frac{(a_n^*)^2 - 1}{(a_n^*)^2}\sigma^2, \quad \text{where} \quad d_3^* = \sqrt{\frac{(a_n^*)^2 - 1}{(a_n^*)^2}} = \sqrt{1 - \frac{1}{(a_n^*)^2}},$$

$$D(u) = \sqrt{V(u)} = d_3^*\,\sigma.$$

< TABLE 3 >

Although statistics $b_n\ R$, $a_n\ s$ are unbiased estimators, R is simpler in calculation than s. As s is calculated by n data and the mean value, it necessarily leads to the difference of efficiency compared to R, which is computed from only two informations of the maximum and the minimum of data. To check this point, we define an index of efficiency as the ratio of $V(a_n\ s)$ to $V(b_n\ R)$, and arrange them in order of n into TABLE 3.

TABLE 3. $\eta(n) = \dfrac{V(a_n\, s)}{V(b_n\, R)}$.

n	$\eta(n)$	n	$\eta(n)$
2	1.0000	11	0.8313
3	0.9919	12	0.8136
4	0.9752	13	0.7968
5	0.9548	14	0.7809
6	0.9330	15	0.7657
7	0.9112	20	0.7002
8	0.8899	30	0.6049
9	0.8695	50	0.4879
10	0.8499	100	0.3477

From TABLE 3 it is noticed that when we use R as an substitute of s, we must set limits to n = 10 in order to keep efficiency of 85% level, or n = 7 to keep that of 90% level.

< TABLE 4 >

A sample with a large number N of data is divided in k groups of the same size n, the standard error of estimation by the mean $\overline{R}$ of all ranges R_i, i = 1, 2, ..., k of groups is given by

$$\sqrt{V(b_n\,\overline{R})} = D(b_n\,\overline{R}) = \frac{1}{\sqrt{k}}\frac{d_3}{d_2}\,\sigma.$$

Similarly, if N is sufficiently large for $N = kn$, it is known to hold

$$\sqrt{V(a_n^*\,u)} = D(a_n^*\,u) \approx \frac{1}{\sqrt{2kn}}\,\sigma.$$

Note. Since

$$c_2{}^* = \sqrt{\frac{2}{N-1}} \cdot \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)},$$

we have

$$c_2{}^* \approx \left(1 - \frac{1}{4N}\right),$$

and

$$c_3{}^* = \sqrt{1-(c_2{}^*)^2} \approx \sqrt{\frac{1}{2N} - \left(\frac{1}{4N}\right)^2} \approx \sqrt{\frac{1}{2N}} = \frac{1}{\sqrt{2N}},$$

for sufficiently large N. Since $N = kn$, we have

$$D\left(\frac{u}{c_2{}^*}\right) \approx \frac{1}{\sqrt{2nk}}\sigma.$$

Therefore, we have

$$\frac{D\left(\frac{\overline{R}}{d_2}\right)}{D\left(\frac{u}{c_2{}^*}\right)} = \sqrt{2\pi}\,\frac{d_3}{d_2}.$$

We arrange the ratio in TABLE 4 by order of n.

TABLE 4. The ratio $D\left(\frac{\overline{R}}{d_2}\right) \Big/ D\left(\frac{u}{c_2{}^*}\right)$.

n	$d_2 = \frac{1}{b_n}$	d_3	$\sqrt{2n}\,\frac{d_3}{d_2}$	n	$d_2 = \frac{1}{b_n}$	d_3	$\sqrt{2n}\,\frac{d_3}{d_2}$
2	1.1284	0.8525	1.5110	11	3.1729	0.7873	1.1639
3	1.6926	0.8884	1.2856	12	3.2585	0.7785	1.1704
4	2.0588	0.8798	1.2087	13	3.3360	0.7705	1.1776
5	2.3259	0.8641	1.1748	14	3.4068	0.7630	1.1852
6	2.5344	0.8480	1.1591	15	3.4718	0.7562	1.1930
7	2.7044	0.8332	1.1528	20	3.7349	0.7287	1.2339
8	2.8472	0.8198	1.1518	30	4.0855	0.6927	1.3133
9	2.9700	0.8078	1.1540	50	4.4981	0.6522	1.4498
10	3.0775	0.7971	1.1583	100	5.0152	0.6052	1.7066

From TABLE 4 the ratio is seen to take the minimum at $n = 8$, i.e., the nearest to the estimation of σ. It takes the best efficiency 86.8% to u for the size $n = 8$.

< TABLE 5 >

We arrange TABLE 5 to show statistical relation between the mean of random variables given by our inequality of the Main Theorem concerning s^2 and R :

$$\sqrt{\frac{1}{2}}\,E(R) \leq \sqrt{n}\,E(s) \leq \sqrt{\frac{n}{4}-\frac{1-(-1)^n}{8n}}\,E(R)\,.$$

From TABLE 5, it is found the difference of $\sqrt{n}\,E(s)$ between the g.l.b and the l.u.b. in its values is larger as n increasing($n>10$). But in $n<10$, the difference seems to be not so far away in comparing the TABLE 3. Therefore, it seems to be possible more consideration and study for us to use another point of view for sample range.

TABLE 5

n	$\sqrt{\frac{1}{2}}E(R)$	$\sqrt{n}\,E(s)$	$\sqrt{\frac{n}{4}-\frac{1-(-1)^n}{8n}}\,E(R)$
2	0.7979	0.7978	0.7978
3	1.1968	1.2533	1.3820
4	1.4558	1.5958	2.0588
5	1.6447	1.8800	2.5479
6	1.7921	2.1277	3.1040
7	1.9123	2.3500	3.5408
8	2.0133	2.5532	4.0265
9	2.1001	2.7416	4.4275
10	2.1761	2.9180	4.8660
20	2.6410	4.3019	8.3516
30	2.8889	5.3389	11.1887
50	3.1807	6.9644	15.9033
100	3.5463	9.9248	25.0759

§ 4. Practical examples

Here we will illustrate our discussion by two examples.

Example 1. Test scores of Mathematics in a high school.

The number of data in the sample is $N = 59$, which is divided in $k = 10$ random groups with the same size $n = 6$ and the rest is 5 data.

TABLE 6

Data						Range (R_i)	Sample s.d. (s)	Mean ($\overline{x}_i$)
57	65	60	62	71	76	19	6.517	65.167
78	68	78	56	61	78	22	8.877	69.833
60	68	70	62	73	62	13	4.775	65.833
53	65	64	63	54	68	15	5.639	61.167
67	62	67	54	67	71	17	5.437	64.667
63	70	63	68	43	66	27	8.933	62.167
73	49	66	61	62	66	24	7.290	62.833
61	59	57	66	71	70	14	5.354	64.000
33	57	65	53	68	70	37	12.539	57.667
48	50	56	67	71		23*	9.135*	58.400
						$\Sigma R_i = 188 + 23$	$s = 8.476$	$\overline{\overline{x}} = 63.254$

By the modification multiplier 0.3946 to each of 9 groups with 6 data and the multiplier 0.4299 to the last group with the rest. Then we have the weighted mean :

$$\overline{R}^* = \frac{188 \times 0.3946 + 23 \times 0.4299}{10} = 8.407.$$

The standard deviation s of the sample with $N = 59$ data is calculated as $s =$ 8.476, and it seems to be same to same approximately. The correlation coefficient of R and s is $r = 0.957$. This is the reason why the estimation $\overline{R}^*$ is near to s. The relative standard error by this estimation method is calculated as 10.6% from $\frac{1}{\sqrt{k}}\frac{d_3}{d_2}$. The correlation of R_i and $\overline{x}$ is given by $r = -0.523$, but it might be no significant for the test with population correlation $\rho = 0$.

Example 2. Test scores of $N = 81$ data divided in $k = 14$ groups with the same size $n = 6$ and the rest is 3 data.

TABLE 7

Data						Range (R_i)	Sample s. d. (s)	Mean $(\overline{x}_i)$
27	43	16	12	8	8	35	12.517	19.000
13	11	53	43	32	10	43	16.823	27.000
10	20	12	29	23	30	20	7.652	20.667
5	53	20	27	25	44	48	15.695	29.000
73	12	30	29	33	80	68	24.815	42.833
55	58	40	80	27	25	55	19.172	47.500
42	18	8	8	42	12	34	14.761	21.667
37	18	44	55	51	22	37	13.849	37.833
32	13	27	44	22	80	67	21.685	36.333
48	35	80	44	80	33	47	19.525	53.333
50	38	85	75	71	50	47	16.540	61.500
65	65	60	70	80	35	45	13.769	62.500
65	70	55	80	95	24	71	22.101	64.833
60	85	20				65*	26.771*	55.000
						$\Sigma R_i = 617 + 65$	$s = 23.864$	$\overline{\overline{x}} = 40.852$

Similar to the previous example, an unbiased estimate $\overline{R}^*$ of σ is given by the weighted mean of the group ranges :

$$\overline{R}^* = \frac{617 \times 0.3946 + 65 \times 0.5908}{14} = 20.134,$$

which is slightly smaller than s. But the relative error is at 9% level. The correlation coefficient of R and s is 0.927, namely, the variation of R is larger, and the correlation of R and s is lower than those of the Example 1. It is because this example may be far from normality and it is noticed that the modification multiplier is calculated under the assumption of normality.

§ 5. Comments

The calculation of sample range is very easy, since it means the difference between the maximum and the minimum data values. But the sample standard deviation is computed from data mean and each of data values, therefore it takes time for sample standard deviation against for sample range. Furthermore, the correlation coefficient between sample ranges and unbiased estimators in each of the divided groups is very high (or equals nearly 1). Therefore R is usable as a substitute of u under certain condition. Because of this reason utilization of R has been promoted in the area of quality control of production process.

Recently in the area of education especially in the compulsory education, the national student achievement assessment is put into operation by the Ministry of Education, Culture, Sports, Science and Technology in Japan, as the interest in the present situation of pupils' or students' scholastic ability has been growing. The aim of the assessment is not to compete for the mean superiority between regional units, but to contribute to the proper measure of education. It means that the method of quality control can be utilized to the student achievement assessment of the compulsory education, and that not the mean score or the sample mean but the dispersion of scores in a certain range from the expected objective level should be investigated. We hope to extend our study for such field in future.

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